On transcendental Progressions or those whose general terms can not be given algebraically *

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§1 After on the occasion of those observations on series Goldbach had discussed in the Society, I recently tried to find a certain general expression, which would give all terms of this progression

$$1+1\cdot 2+1\cdot 2\cdot 3+1\cdot 2\cdot 3\cdot 4+\text{etc.};$$

considering that it, if it is continued to infinity, is finally confounded with the geometric series, I discovered the following expression

$$\frac{1\cdot 2^n}{1+n}\cdot\frac{2^{1-n}\cdot 3^n}{2+n}\cdot\frac{3^{1-n}\cdot 4^n}{3+n}\cdot\frac{4^{1-n}\cdot 5^n}{4+n}\cdot\text{etc.},$$

which expresses the term of order n in the mentioned progression. It certainly does not terminate in any case, neither if n is an integer number nor if it is a fraction, but it only converges to the respective term, if the cases $n=0$ and $n=1$ are excluded, in which the formula is immediately seen to be 1. Put $n=2$; one will then have

$$\frac{2\cdot 2}{1\cdot 2}\cdot\frac{3\cdot 3}{3\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{5\cdot 5}{4\cdot 6}=\text{ the second term 2}$$

If it is $n=3$, one will have

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$$\frac{2\cdot 2\cdot 2}{1\cdot 1\cdot 4}\cdot\frac{3\cdot 3\cdot 3}{2\cdot 2\cdot 5}\cdot\frac{4\cdot 4\cdot 4}{3\cdot 3\cdot 6}\cdot\frac{5\cdot 5\cdot 5}{4\cdot 4\cdot 7}\cdot \text{etc} = \text{to the third term } 6$$

§2 But although this expression seems to have no use for finding the terms, it can nevertheless applied to interpolate the series or the terms, whose indices are fractional numbers. But I decided to explain nothing about this here, since below more appropriate ways to achieve the same will be exhibited. I just want to mention everything, necessary to get to the things to follow, about the general term. I tried to find the term corresponding to the index $n = \frac{1}{2}$ or which falls in the middle between the first, 1, and the preceding one, which is also 1. But having put $n = \frac{1}{2}$ I obtain the product

$$\sqrt{\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot \text{etc.}},$$

which expresses the term in question. But this series to me seemed to be similar to the one I remembered to have seen in Wallis's works on area of the circle. For, Wallis found that the area of the circle has a ratio to the diameter of the circle of

$$2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot \text{etc.} \quad \text{to} \quad 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot \text{etc.}$$

Therefore, if the diameter was $= 1$, the area of the circle will be

$$= \frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot \text{etc.}$$

Therefore, from the agreement of this series with mine it is possible to conclude that the term corresponding to the index $\frac{1}{2}$ is equal to the square root of the circle, whose diameter is $= 1$.

§3 I had believed before that the general term of the series 1, 2, 6, 24 etc., if not algebraically, is at least given exponentially. But after I had understood that certain intermediate terms depend on the quadrature of the circle, I realized that neither algebraic nor exponential quantities are sufficient to express it. For, the general term of the progression must be of such a nature that the one time it contains algebraic quantities the other time quantities depending on the quadrature of the circle and even on other quadratures; and no algebraic nor exponential formula fulfills this condition.

§4 But after I had remembered that among differential formulas quantities exist, which in certain cases admit an integration and then yield algebraic quantities, but in other cases do not admit an integration and then exhibit quantities depending on the quadratures of curves, it came to mind that maybe formulas of this kind are appropriate to express the general terms of the mentioned progression and other similar ones. And in the following I will call progressions requiring such general terms, which cannot be given algebraically, *transcendental*; as Geometers used to call everything exceeding the power of common Algebra, transcendental.

§5 Therefore, I thought about, what properties differential formulas should have in order to express general terms of progressions the best way possible. But the general term is a formula, into which so constant as certain other non constant quantities as n enter, which number n gives the order of the terms or the index that, if the third term is in question, instead of n one has to put 3. But also a certain variable quantity must be contained in the differential formula. For this it is not advisable to use n, since this variable is not to be integrated over, but, after the formula had been integrated or is assumed that it had been integrated, this variable n is just used to form the progression. Therefore, a certain variable quantity x must be contained in the differential formula, which after the integration is to be put equal to another number concerning the progression[1]; and hence the term, whose index is n, results.

§6 That this is understood more clearly, I say that $\int pdx$ is the general term of the progression to be found in the following from it; but let p denote an arbitrary function of x and constants, one of which constants must be n. Now assume pdx to be integrated and augmented by such a constant that having put $x = 0$ the whole integral vanishes; then put x equal to a certain known quantity. Having done this only quantities related to the progression will remain in the found integral, and that integral will express the term corresponding to the index $= n$. Or the integral determined in this way will be the general term. If this integration is actually possible, the differential formula in not necessary and the progression formed from this will have a general algebraic term; but matters change, if the integration only succeeds for certain numbers substituted for n.

[1] By this Euler means that the upper limit of the integral has to be chosen appropriately

§7 Therefore, I considered and tried many differential formulas of this kind only admitting an integration, if a positive integer is substituted for n, so that the principal terms become algebraic, and hence formed progressions. Therefore, their general terms were immediately clear, and it will be possible to define the quadrature describing the intermediate terms. I will certainly not go through many formulas of this kind here, but will only treat a general one extending very far which can be accommodated to all progressions, whose arbitrary terms are products consisting of a certain number of terms depending on the index; the factors of these products are fractions, whose numerators and denominators proceed in an arbitrary arithmetic progression, as, e.g.,

$$\frac{2}{3}+\frac{2\cdot 4}{3\cdot 5}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}+\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}+\text{ etc.}$$

§8 Let this formula be propounded

$$\int x^e dx(1-x)^n$$

describing the general term; this formula integrated in such a way that it becomes $=0$, if it is $x=0$, and then having put $x=1$ then expresses the term of order n of the progression resulting from it. Therefore, let us see, which progression it actually describes. It is

$$(1-x)^n=1-\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\text{ etc.}$$

and hence

$$dx(1-x)^n=x^e dx-\frac{n}{1}x^{e+1}dx+\frac{n(n-1)}{1\cdot 2}x^{e+2}dx-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{e+3}dx+\text{ etc.}$$

Hence

$$\int x^e dx(1-x)^n=\frac{x^{e+1}}{e+1}-\frac{nx^{e+2}}{1\cdot(e+2)}+\frac{n(n-1)x^{e+3}}{1\cdot 2\cdot(e+3)}-\frac{n(n-1)(n-2)x^{e+4}}{1\cdot 2\cdot 3\cdot(e+4)}+\text{ etc.}$$

Put $x=1$, since the addition of the constant it not necessary, and one will have

$$\frac{1}{e+1}-\frac{n}{1\cdot(e+2)}+\frac{n(n-1)}{1\cdot 2\cdot(e+3)}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot(e+4)}+\text{ etc.}$$

as the general term in question of the series. The series will be of such a nature that, if it is $n = 0$, the corresponding term results to be $= \frac{1}{e+1}$; if $n = 1$, the term $\frac{1}{(e+1)(e+2)}$; if $n = 2$, the corresponding term is $= \frac{1 \cdot 2}{(e+1)(e+2)(e+3)}$, if it is $n = 3$, then the corresponding term is $= \frac{1 \cdot 2 \cdot 3 \cdot 4}{(e+1)(e+2)(e+3)(e+4)}$; the rule describing how these terms proceed, is manifest.

§9 Therefore, I obtained this progression

$$\frac{1}{(e+1)(e+2)} + \frac{1 \cdot 2}{(e+1)(e+2)(e+3)} + \frac{1 \cdot 2 \cdot 3}{(e+1)(e+2)(e+3)(e+4)} + \text{ etc.},$$

whose general term is

$$\int x^e dx (1-x)^n.$$

On the other hand the term of order n will be expressed by this form

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{(e+1)(e+2) \cdots (e+n+1)}.$$

This form certainly suffices to find the terms of integer indices, but if the indices were no integers, this form cannot be used to find the corresponding terms. This following series will be helpful to find them approximately

$$\frac{1}{e+1} - \frac{n}{1 \cdot (e+2)} + \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot (e+4)} + \text{ etc.}$$

If $\int x^e dx (1-x)^n$ is multiplied by $e+n+1$, one will have a progression, whose term of order n has this form

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(e+1)(e+2) \cdots (e+n)},$$

whose general term will therefore be

$$(e+n+1) \int x^e dx (1-x)^n.$$

Here it is to be noted that the progression always becomes algebraic, whenever a positive number is assumed for e. Put, e.g., $e = 2$, then the n-th term of the progression will be

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{3 \cdot 4 \cdot 5 \cdots (n+2)} \quad \text{or} \quad \frac{1 \cdot 2}{(n+1)(n+2)}$$

The general term itself also indicates this; for, this term will be

$$(n+3)\int xxdx(1-x)^n.$$

For, its integral is

$$\left(C-\frac{(1-x)^{n+1}}{n+1}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+3}}{n+3}\right)(n+3);$$

in order for this to become $=0$, if it is $x=0$, it will be

$$C=\frac{1}{n+1}-\frac{2}{n-2}+\frac{1}{n+3}.$$

Put $x=1$; the general term will be

$$\frac{n+3}{n+1}-\frac{2(n+3)}{n+2}+1=\frac{2}{(n+1)(n+2)}.$$

§10 Therefore, in order to obtain transcendental progressions, put e equal to the fraction $\frac{f}{g}$. The term of order n of the progression will be

$$=\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}g^n$$

or

$$\frac{g\cdot 2g\cdot 3g\cdots ng}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

The general term on the other hand will be

$$=\frac{f+(n+1)g}{g}\int x^{\frac{f}{g}}dx(1-x)^n.$$

If this is divided by g^n, we will obtain the general term for the progression

$$\frac{1}{f+g}+\frac{1\cdot 2}{(f+g)(f+2g)}+\frac{1\cdot 2\cdot 3}{(f+g)(f+2g)(f+3g)}+\text{ etc.},$$

whose term of n-th order is

$$=\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)}.$$

Therefore, the general term of the progression will be

$$= \frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx (1-x)^n$$

If here the fraction $\frac{f}{g}$ is not equal to an integer number, or if f is no multiple of g, the progression will be transcendental and the intermediate terms will depend on quadratures.

§11 In order to display the general term more clearly, I want to mention a certain example. In the first progression of the preceding paragraph let $f = 1$ and $g = 2$; the term of order n will be

$$= \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)},$$

the progression itself on the other hand will be this one

$$\frac{2}{3} + \frac{2 \cdot 4}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} + \text{ etc.},$$

whose general term will hence be

$$\frac{2n+3}{2} \int dx (1-x)^n \sqrt{x}.$$

Let the term corresponding to the index $\frac{1}{2}$ be in question; therefore, it will be $n = \frac{1}{2}$ and one will find the general term in question to be

$$= 2 \int dx \sqrt{x - xx}$$

Since this denotes the element of the circular area, it is perspicuous that the term in question is the area of the circle, whose diameter is $= 1$.
Further, let this series be propounded

$$1 + \frac{r}{1} + \frac{r(r-1)}{1 \cdot 2} + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} + \text{ etc.},$$

which is the coefficient of the binomial raised to the power r. Therefore, the term of order n is

$$\frac{r(r-1)(r-2) \cdots (r-n+2)}{1 \cdot 2 \cdot 3 \cdots (n-1)}.$$

In the preceding paragraph on the other hand we found this expression

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)}.$$

In order to compare these two expressions, invert the second one so that one has

$$\frac{(f+g)(f+2g)\cdots(f+ng)}{1\cdot 2\cdots n};$$

now multiply it by $\frac{n}{f+ng}$ and it will be

$$=\frac{(f+g)(f+2g)\cdots(f+(n-1)g)}{1\cdot 2\cdots(n-1)};$$

therefore, it must be $f+g=r$ and $f+2g=r-1$, whence it will be $g=-1$ and $f=r+1$. Treat the following general term in the same way

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

For the propounded progression

$$1+\frac{r}{1}+\frac{r(r-1)}{1\cdot 2}+\text{ etc.}$$

this general term will result

$$\frac{n(-1)^{n+1}}{(r-n)(r-n+1)\int x^{-r-1}dx(1-x)^n}.$$

Let $r=2$; the general term of this progression

$$1,\quad 2,\quad 1,\quad 0,\quad 0,\quad 0\text{etc.}$$

will be

$$\frac{n(-1)^{n+1}}{(2-n)(3-n)\int x^{-3}dx(1-x)^n}.$$

Here it must be noted that this and other cases, in which $e+1$ is a negative number, cannot be deduced from the general expression; for, in these cases the integral does not become $=0$, if it is $x=0$. But in order to treat also these cases is convenient to integrate

$$\int x^e dx(1-x)^n$$

in a peculiar way; for, after the integration an infinite constant is to be added. But whenever $e+1$ is a positive number, as I assumed in § 8, the addition of the constant is not necessary. But having considered the progression, whose term of order n was the following

$$\frac{r(r-1)(r-2)\cdots(r-n+2)}{1\cdot 2\cdot 3\cdots(n-1)},$$

that form of the term corresponding to the index n can be transformed into this one

$$\frac{r(r-1)\cdots 1}{(1\cdot 2\cdot 3\cdots(n-1))(1\cdot 2\cdots(r-n+1))}.$$

But by means of § 14 it is

$$r(r-1)\cdots 1=\int dx(-\ln(x))^r$$

and

$$1\cdot 2\cdot 3\cdots(n-1)=\int dx(-\ln(x))^{n-1}$$

and

$$1\cdot 2\cdots(r-n+1)=\int dx(-\ln(x))^{r-n+1}.$$

Therefore, the progression considered there

$$1+\frac{r}{1}+\frac{r(r-1)}{1\cdot 2}+\frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}+\text{ etc.}$$

has this general term

$$\frac{\int dx(-\ln(x))^r}{\int dx(-\ln(x))^{n-1}\int dx(-\ln(x))^{r-n+1}}-$$

If it was $r=2$, the general term will be

$$\frac{2}{\int dx(-\ln(x))^{n-1}\int dx(-\ln(x))^{3-n}},$$

to which this progression corresponds

$$1,\quad 2,\quad 1,\quad 0,\quad 0,\quad 0\text{ etc.};$$

as if the term of the index $\frac{3}{2}$ is in question, it will be

$$\frac{2}{\int dx(-\ln(x))^{\frac{1}{2}}\int dx(-\ln(x))^{\frac{3}{2}}}.$$

Therefore, having called the area of the circle, whose diameter is $=1$, A, since it is

$$\int dx(-\ln(x))^{\frac{1}{2}}=\sqrt{A} \quad \text{and} \quad \int dx(-\ln(x))^{\frac{3}{2}}=\frac{3}{2}\sqrt{A},$$

the term falling in the middle between the first two terms of the progression 1, 2, 1, 0, 0, 0 etc. will be $\frac{4}{3A}$, this means approximately $\frac{5}{3}$.

§12 Now, I proceed to the progression I talked about at the beginning,

$$1+1\cdot 2+1\cdot 2\cdot 3+ \text{ etc.}$$

and in which the term corresponding to the index n is $1\cdot 2\cdot 3\cdot 4\cdots n$. This progression is contained in our general one, but the general term must be derived from it in a peculiar way. Until now I considered the general term, if the term of order n is

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)\cdots(f+ng)},$$

which, if one puts $f=1$ and $g=0$, goes over into $1\cdot 2\cdot 3\cdots n$, which is in question; therefore, in the general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n$$

substitute these values for f and g; the general term in question will be

$$\int\frac{x^{\frac{1}{0}}dx(1-x)^n}{0^{n+1}}.$$

And I will investigate value of this general term as follows.

§13 Considering the condition, general terms of this kind must fulfill in order to be useful, it is understood that instead of x other functions can be assumed, as long as they were of such a kind that they are $=0$, if it is $x=0$, and $=1$, if $x=1$. For, if a function of this kind is substituted for x, the general term will therefore fulfill the same condition as before. Therefore, put $x^{\frac{p}{f+g}}$

instead of x and as a logical consequence $\frac{g}{f+g}x^{\frac{-f}{g+f}}dx$ instead of dx; having done this one will have

$$\frac{f+(n+1)g}{g^{n+1}}\int\frac{g}{f+g}dx(1-x^{\frac{g}{f+g}})^n.$$

Now put $f=1$ and $g=0$ here; one will have

$$\int\frac{dx(1-x^n)}{0^n}.$$

But because it is $x^0=1$, here we have a case, in which the numerator $(1-x^0)^n$ and the denominator 0^n vanish. Therefore, applying the known rule[2] let us find the value of the fraction $\frac{1-x^0}{0}$. This will by achieved by finding the value of the fraction $\frac{1-x^z}{z}$ for vanishing z; therefore, differentiate the numerator and the denominator with respect to that variable z; one will find $\frac{-x^z dt\ln(x)}{dz}$ or $-x^z\ln(x)$; if now one puts $z=0$, $-\ln(x)$ will result. Therefore, it is

$$\frac{1-x^0}{0}=-\ln(x).$$

§14 Therefore, because it is

$$\frac{1-x^0}{0}=-\ln(x),$$

it will be

$$\frac{(1-x^0)^n}{0^n}=(-\ln(x))^n$$

and therefore the general term in question $\int\frac{dx(1-x^0)^n}{0^n}$ is transformed into $\int dx(-\ln(x))^n$. Its value can be expressed by means of quadratures. Therefore, the general term of this progression

$$1,\quad 2,\quad 6,\quad 24,\quad 120,\quad 720\quad \text{etc.}$$

is

$$\int dx(-\ln(x))^n,$$

[2]By this Euler means L'Hôpital's rule.

and it is to be used in the same way as it was prescribed above. That this is really the general term of the propounded progression is also seen from this, that it indeed yields the correct terms for the cases, in which the indices are positive integers. Let, for the sake of an example, it be $n = 3$; it will be

$$\int dx(-\ln(x))^3 = \int -dx(\ln(x))^3 = -x(\ln(x))^3 + 3x(\ln(x))^2 - 6x\ln(x) + 6x;$$

the addition of a constant is not necessary, since for $x = 0$ everything vanishes; therefore, put $x = 1$; since $\ln(1) = 0$, all terms containing which logarithms will vanish and 6 will remain, which is the third term.

§15 It is clear that this method to find the terms of this series is too laborious, even for those terms corresponding to integer indices; they are certainly obtained easier by just continuing the progression. But this expression is nevertheless more than appropriate to find the terms corresponding to rational indices; for, it was not even possible to define these terms using the most laborious methods. If one puts $x = \frac{1}{2}$, the corresponding term will be $= \int dx\sqrt{-\ln(x)}$, whose value is given by quadratures. But at the beginning [§ 11] I showed that this term is equal to the square root of the area the circle, whose diameter is 1. Hence it is certainly not possible to conclude the same because of the missing Analysis; but below a method will be explained to reduce the same intermediate terms to quadratures of algebraic curves. And comparing this method to the one already explained it will maybe possible to derive many results which can be used to develop the whole field of Analysis even further.

§16 The general term of the progression, whose term of order n is given by

$$\frac{1 \cdot 2 \cdot 3 \cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)},$$

by means of § 10 is

$$\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx(1-x)^n.$$

But if the term of order n was

$$1 \cdot 2 \cdot 3 \cdots n,$$

then the general term is

$$\int dx(-\ln(x))^n.$$

If this formula is substituted for $1 \cdot 2 \cdot 3 \cdots n$, one will have

$$\frac{\int dx(-\ln(x))^n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx(1-x)^n.$$

Hence it is

$$(f+g)(f+2g)(f+3g)\cdots(f+ng) = \frac{g^{n+1}\int dx(-\ln(x))^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n}.$$

Therefore, this expression is the general term of this general progression

$$(f+g) \quad ,(f+g)(f+2g) \quad ,(f+g)(f+2g)(f+3g) \quad \text{etc.}$$

Therefore, by means of the general term all terms corresponding to any arbitrary index of all progressions of this kind are defined. What will follow below about the reduction of $\int dx(-\ln(x))^n$ to more known quadratures or quadratures of algebraic curves, will also be useful here.

§17 Let $f+g=1$ and $f+2g=3$; it will be $g=2$ and $f=-1$. Hence this particular progression will result

$$1, \quad 1\cdot 3, \quad 1\cdot 3\cdot 5, \quad 1\cdot 3\cdot 5\cdot 7\text{etc.}$$

Therefore, its general term is

$$\frac{2^{n+1}\int dx(-\ln(x))^n}{(2n+1)\int x^{-\frac{1}{2}}dx(1-x)^n}.$$

Although here one exponent of x is negative, nevertheless the inconvenience addressed above does not occur here, since it is greater than -1. Put $n=\frac{1}{2}$ to find the term corresponding to the index $\frac{1}{2}$; that term will be

$$= \frac{2^{\frac{3}{2}}\int dx\sqrt{-\ln(x)}}{2\int x^{-\frac{1}{2}}dx\sqrt{1-x}} = \frac{\sqrt{2}\int dx\sqrt{-\ln(x)}}{\int \frac{dx-xdx}{\sqrt{x-xx}}}.$$

But from § 15 it is known that $\int dx\sqrt{-\ln(x)}$ gives the square root of the circle, whose diameter is $=1$; let the circumference of that circle be p; the area will be $=\frac{1}{4}p$ and hence $\int dx\sqrt{-\ln(x)}$ gives $\frac{1}{2}\sqrt{p}$. Further,

$$\int \frac{dx-xdx}{2\sqrt{x-xx}}=\int \frac{dx}{2\sqrt{x-xx}}+\sqrt{x-xx},$$

but $\int \frac{dx}{2\sqrt{x-xx}}$ gives the arc, whose sinus versus is x, of the circle. Therefore, having put $x=1$ $\frac{1}{2}p$ will result. Therefore, the term in question will be

$$=\sqrt{\frac{2}{p}}.$$

§18 Since the general term of the progression, whose term of order n is given by

$$(f+g)(f+2g)\cdots(f+ng),$$

by means of § 16 is

$$\frac{g^{n+1}\int dx(-\ln(x))^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n},$$

similarly, if the term of order n was

$$(h+k)(h+2k)\cdots(h+nk),$$

the general term will be

$$\frac{k^{n+1}\int dx(-\ln(x))^n}{(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}.$$

Divide that progression by this one, namely the first term by the first, the second by the second and so forth; this way one will get to a new progression, whose term of order n will be

$$\frac{(f+g)(f+2g)\cdots(f+ng)}{(h+k)(h+2k)\cdots(h+nk)}.$$

And the general term of this progression composited of these two will be

$$\frac{g^{n+1}(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}{k^{n+1}(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n}.$$

This term does not contain the logarithmic integral $\int dx(-\ln(x))^n$

§19 In all general terms of this kind it is especially to be noted that not even for f, g, h, k one has to put constant numbers, but they can be assumed to depend on n arbitrarily. For, in the integration these letters are treated in the same way as n, namely as constants. Therefore, let the term of order n be this one

$$(f+g)(f+2g)\cdots(g+ng);$$

put $g=1$, but $f=\frac{nn-n}{2}$. Since the progression itself is

$$f+g, \quad (f+g)(f+2g), \quad (f+g)(f+2g)(f+3g) \quad \text{etc.}$$

put 1 instead of g everywhere; the progression will be

$$f+1, \quad (f+1)(f+2), \quad (f+1)(f+2)(f+3) \quad \text{etc.}$$

But one has to write 0 in the first term instead of f, 1 in the second, 3 in the third, 6 in the fourth and so forth; then this progression will result

$$1, \quad 1\cdot 2, \quad 4\cdot 5\cdot 6, \quad 7\cdot 8\cdot 9\cdot 10 \quad \text{etc.},$$

whose general term is

$$\frac{2\int dx(-\ln(x))^n}{(nn+n+2)\int x^{\frac{nn-n}{2}}dx(1-x)^n} = \frac{2\int dx(-\ln(x))^n}{(nn+n+2)\int dx(x^{\frac{n-1}{2}}-x^{\frac{n+1}{2}})^n}$$

§20 Now I proceed to the progressions, which led me to the invention of the artifice to define the intermediate of this progression more easily

$$1, \quad 2, \quad 6, \quad 24, \quad 120 \quad \text{etc.}$$

For, this artifice extends further than only to this progression, since its general term

$$\int dx(-\ln(x))^n$$

also enters the general terms of infinitely many other progressions. I assume this general term

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n,$$

to which this term of order n corresponds

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots(f+ng)}.$$

Here I put $f=n$, $g=1$; hence the general term will be

$$(2n+1)\int x^n dx(1-x)^n \quad \text{or} \quad (2n+1)\int dx(x-xx)^n$$

and its form of order n will be

$$\frac{1\cdot 2\cdot 3\cdots n}{(f+g)(f+2g)(f+3g)\cdots 2n}.$$

The progression itself on the other hand is this one

$$\frac{1}{2},\quad \frac{1\cdot 2}{3\cdot 4},\quad \frac{1\cdot 2\cdot 3}{4\cdot 5\cdot 6}\quad \text{etc.}$$

or this one

$$\frac{1\cdot 1}{1\cdot 1},\quad \frac{1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4},\quad \frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}.$$

In this expression the numerators are the squares of the progression 1, 2, 6, 24 etc.; and now it is easily to find the terms corresponding to rational indices whose denominator is 2. For, in the progression 1, 2, 6, 24 etc. let the term corresponding to the index $\frac{1}{2}$ be A; the term of order $\frac{1}{2}$ of the propounded progression will then be $=\frac{AA}{1}$.

§21 In the general term

$$(2n+1)\int x^n dx(1-x)^n$$

put $n=\frac{1}{2}$; the term corresponding to that exponent will be

$$2\int dx\sqrt{x-xx}=\frac{AA}{1},$$

whence

$$A = \sqrt{1 \cdot 2 \int dx \sqrt{x - xx}}$$

= to the term of the progression 1, 2, 6, 24 etc. corresponding to the index $\frac{1}{2}$; hence this term, as it is clear from the integral, is the square root of the area of the circle, whose diameter is 1. Now call the term of order $\frac{3}{2}$ of this progression A; the corresponding term in the assumed progression will be

$$\frac{A \cdot A}{1 \cdot 2 \cdot 3} = 4 \int dx (x - xx)^{\frac{3}{2}},$$

therefore

$$A = \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \int dx (x - xx)^{\frac{3}{2}}}.$$

In like manner the term of order $\frac{5}{2}$ will be found to be

$$= \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \int dx (x - xx)^{\frac{5}{2}}}.$$

From these I conclude in general that the term of order $\frac{p}{2}$ will be

$$= \sqrt{1 \cdot 2 \cdot 3 \cdot 4 \cdots (p+1) \int dx (x - xx)^{\frac{p}{2}}}.$$

Therefore, this way one finds all terms of the progression 1, 2, 6, 24 etc., whose indices are fractions, while the denominator is 2.

§22 Further, in the general term

$$\frac{f + (n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx (1 - x)^n$$

I put $f = 2n$, while g remains $= 1$; then this expression results

$$(3n+1) \int dx (xx - x^3)^n$$

as general term of the progression

$$\frac{1}{3}, \quad \frac{1 \cdot 2}{5 \cdot 6}, \quad \frac{1 \cdot 2 \cdot 3}{7 \cdot 8 \cdot 9} \quad \text{etc.}$$

Multiply that one by the preceding $(2n+1) \int dx (x - xx)^n$; then this expression will result

$$(2n+1)(3n+1)\int dx(x-xx)^n \int dx(xx-x^3)^n.$$

This will give this progression

$$\frac{1\cdot 1\cdot 1}{1\cdot 2\cdot 3}, \frac{1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} \quad \text{etc.,}$$

where the numerators are the cubes of the corresponding terms of the progression 1, 2, 6, 24 etc. Let the term of order $\frac{1}{3}$ of this progression be A; the corresponding term of that progression will be

$$\frac{A^3}{1} = 2\left(\frac{2}{3}+1\right)\int dx(x-xx)^{\frac{1}{3}} \int dx(xx-x^3)^{\frac{1}{3}},$$

therefore, the term of order $\frac{1}{3}$ is

$$\sqrt[3]{1\cdot 2\cdot \frac{5}{3}\int dx(x-xx)^{\frac{1}{3}} \int dx(xx-x^3)^{\frac{1}{3}}} :$$

similarly the term of order $\frac{2}{3}$ is

$$\sqrt[3]{1\cdot 2\cdot 3\cdot \frac{7}{3}\int dx(x-xx)^{\frac{2}{3}} \int dx(xx-x^3)^{\frac{2}{3}}}.$$

And the term of order $\frac{4}{3}$ is

$$\sqrt[3]{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \frac{11}{3}\int dx(x-xx)^{\frac{4}{3}} \int dx(xx-x^3)^{\frac{4}{3}}}$$

and in general the term of $\frac{p}{3}$ is

$$\sqrt[3]{1\cdot 2\cdots p\cdot \frac{2p+3}{3}\cdot (p+1)\int dx(x-xx)^{\frac{p}{3}} \int dx(xx-x^3)^{\frac{p}{3}}}.$$

§23 If we want to proceed further by putting $f=3n$, it will be necessary to multiply the general term

$$(4n+1)\int dx(x^3-x^4)^n$$

by the preceding ones, whence one has

$$(2n+1)(3n+1)(4n+1)\int dx(x-xx)^n \int dx(x^2-x^3)^n,$$

which is the general term of this series

$$\frac{1\cdot 1\cdot 1\cdot 1}{1\cdot 2\cdot 3\cdot 4}, \quad \frac{1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2\cdot 1\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} \quad \text{etc.}$$

Using this expression the terms of the progression 1, 2, 6, 24 etc. will be defined, whose indices are fractions having the denominator 4. For, the term, whose index is $\frac{p}{4}$, will be found to be

$$= \sqrt[4]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{4}+1\right)\left(\frac{3p}{4}+1\right)(p+1)}$$
$$\times \int dx(x-xx)^{\frac{p}{4}} \int dx(xx-x^3)^{\frac{p}{4}} \int dx(x^3-x^4)^{\frac{p}{4}}.$$

Hence it is possible to conclude in general that the term of order $\frac{p}{q}$ is

$$= \sqrt[q]{1\cdot 2\cdot 3\cdots p\left(\frac{2p}{q}+1\right)\left(\frac{3p}{q}+1\right)\left(\frac{4p}{q}+1\right)\cdots(p+1)}$$
$$\times \int dx(x-xx)^{\frac{p}{q}} \int dx(x^2-x^3)^{\frac{p}{q}} \int dx(x^3-x^4)^{\frac{p}{q}}\cdots \int dx(x^{q-1}-x^q)^{\frac{p}{q}}.$$

Therefore, from this formula the terms corresponding to any arbitrary fractional indices are found by means of a quadrature of algebraic curves; but for this $1\cdot 2\cdot 3\cdots p$ is required, the term, whose index is the numerator of the propounded fraction.

§24 In the same way it is possible to proceed further to higher composited progressions by assuming higher composited numbers, but I will not persecute this any further here. It is also possible to use multiple integrals so that the general term is

$$\int qdx \int pdx;$$

for, the integral of pdx must be multiplied by the qdx and what results from the integration must be then integrated again; and the result of this second integration will just then having put $x=1$ give the general term of the series. But in each of the two integrations, in order for it to be well-defined, a constant of such a kind has to be added that having put $x=0$ the integral likewise becomes $=0$.

In like manner general terms are to be treated, which are expressed using several integrals, as

$$\int r dx \int q dx \int p dx$$

But nevertheless instead of p, q, r always functions of such a kind are to be taken, that, as often as n was a positive integer, at least algebraic terms result.

§25 Let the general term be

$$\int \frac{dx}{x} \int x^e dx (1-x)^n;$$

this expression converted into a series gives

$$\frac{x^{e+1}}{(e+1)^2} - \frac{nx^{e+2}}{1 \cdot (e+2)^2} + \frac{n(n-1)x^{e+3}}{1 \cdot 2 \cdot (e+3)^2} - \text{etc.}$$

Having put $x = 1$ one will have the term of order n expressed by this series

$$\frac{1}{(e+1)^2} - \frac{n}{1 \cdot (e+2)^2} + \frac{n(n-1)}{1 \cdot 2 \cdot (e+3)^2} - \text{etc.}$$

The progression itself will be this, beginning from the term corresponding to the index 0,

$$\frac{1}{(e+1)^2}, \quad \frac{(e+2)^2 - (e+1)^2}{(e+2)^2(e+1)^2}, \quad \frac{(e+3)^2(e+2)^2 - 2(e+3)^2(e+1)^2 + (e+2)^2(e+1)^2}{(e+3)^2(e+2)^2(e+1)^2}$$

$$\frac{(e+4)^2(e+3)^2(e+2)^2 - 3(e+4)^2(e+3)^2(e+1)^2 + 3(e+4)^2(e+2)^2(e+1)^2 - (e+3)^2(e+2)^2(e+1)^2}{(e+4)^2(e+3)^2(e+2)^2(e+1)^2}$$

etc.

The structure of this progression is manifest and does not require any explanation. Let $e = 0$; it will be

$$\int dx(1-x)^n = \frac{1-(1-x)^{n+1}}{n+1};$$

the general term therefore is

$$\int \frac{dx - dx(1-x)^{n+1}}{(n+1)x},$$

the progression on the other hand will be this one

$$\frac{1}{1}, \quad \frac{4-1}{4\cdot 1}, \quad \frac{9\cdot 4 - 2\cdot 9\cdot 1 + 4\cdot 1}{9\cdot 4\cdot 1}, \quad \frac{16\cdot 9\cdot 4 - 3\cdot 16\cdot 9\cdot 1 + 3\cdot 16\cdot 4\cdot 1 - 9\cdot 4\cdot 1}{16\cdot 9\cdot 4\cdot 1}.$$

The difference will constitute this progression

$$\frac{-1}{4\cdot 1}, \quad \frac{-9+4}{9\cdot 4\cdot 1}, \quad \frac{-16\cdot 9 + 2\cdot 16\cdot 4 - 9\cdot 4}{16\cdot 9\cdot 4\cdot 1} \text{etc.}$$

§26 Therefore, in this dissertation I achieved this, what I mainly intended, namely to I find the general terms of all progressions, whose single terms are products of factors proceeding in an arithmetic progression, and in which the number of factors depends on the index in an arbitrary manner. But although here the number of factors is always put equal to the index, if the number of factors is desired to depend on it in another way, this will not cause any difficulties. The index is denoted by the letter n; if now anyone would require that the number of factors is $\frac{nn+n}{2}$, it is only necessary to substitute $\frac{nn+n}{2}$ for n everywhere.

§27 Instead of ending the dissertation here I want to add something more curious than useful. It is known that by $d^n x$ the differential of order n of x is understood and $d^n p$, if p denotes a certain function of x and dx is put constant, is proportialnal to dx^n; and, if n is an positive integer number, the ratio of $d^n p$ to $d^n x$ can always be expressed algebraically; consider, e.g., the case $n = 2$ and $p = x^3$, the ratio of $d^2(x^3)$ to dx^2 will be the same as $6x$ to 1. Now it is in question, what that ratio will be, if n is a rational number. The difficulty is easily understood in these cases; for, if n is a positive integer number, d^n is found by iterated differentiation; but this is not possible, if n is a fractional number. But nevertheless by means of interpolations of the progressions I considered in this dissertation it will be possible to answer this question.

§28 Let the ratio of $d^n(z^e)$ to dz^n to be found for constant dz, or let the value of the fraction $\frac{d^n(z^e)}{dz^n}$ be in question. First, let us see,what its values are, if n is an integer number; after this we will then proceed to the non-integer cases. If it is $n = 1$, its value will be

$$ez^{e-1} = \frac{1\cdot 2\cdot 3\cdots e}{1\cdot 2\cdot 3\cdots (e-1)} z^{e-1};$$

I express e this way, that later the results we found in this paper can be applied more easily here.
If it is $n = 2$, the value will be

$$e(e-1)z^{e-2} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-2)} z^{e-2}.$$

If it is $n = 3$, one will have

$$e(e-1)(e-2)z^{e-3} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-3)} z^{e-3}.$$

Hence, I generally infer, whatever n is, that it will always be

$$\frac{d^n(z^e)}{dz^n} = \frac{1 \cdot 2 \cdot 3 \cdots e}{1 \cdot 2 \cdot 3 \cdots (e-n)} z^{e-n}.$$

But by means of § 14 it is

$$1 \cdot 2 \cdot 3 \cdots e = \int dx(-\ln(x))^e \text{ und } 1 \cdot 2 \cdot 3 \cdots (e-n) = \int dx(-\ln(x))^{e-n}.$$

Hence one has

$$\frac{d^n(z^e)}{dz^n} = z^{e-n} \frac{\int dx(-\ln(x))^e}{\int dx(-\ln(x))^{e-n}}$$

or

$$d^n(z^e) = z^{e-n} dz^n \frac{\int dx(-\ln(x))^e}{\int dx(-\ln(x))^{e-n}}$$

Here dz is put constant and $\int dx(-\ln(x))^s$ and $\int dx(-\ln(x))^{s-n}$ must be integrated in such a way as it was prescribed above[3].

§29 It is not necessary to show, how the true value is found; this will become clear by substituting an arbitrary integer number for n. But let it be in question what $d^{\frac{1}{2}}z$ is, if dz is constant. Therefore, it will be $e = 1$ and $n = \frac{1}{2}$ in our expression. Therefore, one will have

$$d^{\frac{1}{2}}z = \frac{\int dx(-\ln(x))}{\int dx\sqrt{-\ln(x)}} \sqrt{zdz}.$$

[3]Namely that the limits of integration are $x = 0$ and $x = 1$.

But it is

$$\int dx(-\ln(x)) = 1$$

and having called the area of the circle, whose diameter is 1, A, it will be

$$\int dx\sqrt{-\ln(x)} = \sqrt{A},$$

whence it is

$$d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}.$$

Therefore, let this equation for a certain curve be propounded

$$yd^{\frac{1}{2}}z = z\sqrt{dy},$$

where dz is put constant, and let it be in question, what kind of curve this is. Since it is $d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}$, the equation will go over into this one

$$y\sqrt{\frac{zdz}{A}} = z\sqrt{dz},$$

which squared gives

$$\frac{yydz}{A} = zdy,$$

whence one finds

$$\frac{1}{A}\ln(z) = C - \frac{1}{y}$$

or

$$y\ln(z) = cAy - A,$$

which is the equation for the curve in question.